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Some Physical Implications of a New Relativistic Boundary Condition

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Abstract

It is shown that a new boundary condition in general relativity can be interpreted as a condition on the rate of spinning in a model for the gravitational field of an isolated body embedded in Trautman's expanding universe of spinning particles. The new condition is also shown to be independent of the usual O'Brien–Synge conditions in the sense that it is not an identity following from them.

The validity of the delta function technique employed in deriving the above new boundary condition is investigated in a non-relativistic framework. The technique is shown to yield familiar non-relativistic results as well as a new one which involves rate-of-strain and pressure-gradient in the case of an adiabatic flow such as in a compressible fluid with an isentropic equation.

1. *Introduction*

The problem of formulating suitable boundary conditions in general relativity has received considerable attention over the years.

By introducing a system of coordinates x^i relative to which the non-null boundary surface $\Sigma : x^0 = 0$ is at rest (where x^0 could be any x^i, $i = 1, 2, 3, 4$), and by defining any discontinuity of the energy-momentum tensor T^i_j on Σ as a suitable limit of some continuous distribution of T^i_j, O'Brien & Synge (1952) derived the following boundary conditions:

$$[\overset{(\mathrm{I})}{g_{ij}}] = [\overset{(\mathrm{II})}{g_{ij}}] \tag{1.1}$$

$$[\overset{(\mathrm{I})}{g_{\alpha\beta,0}}] = [\overset{(\mathrm{II})}{g_{\alpha\beta,0}}], \qquad (\alpha, \beta \neq 0) \tag{1.2}$$

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$$\overset{(\mathrm{I})}{[T_i^0]} = \overset{(\mathrm{II})}{[T_i^0]} \tag{1.3}$$

$$[\overset{(\mathrm{I})}{g}_{\alpha i}\overset{(\mathrm{I})}{T_j^\alpha} - \overset{(\mathrm{I})}{g}_{\alpha j}\overset{(\mathrm{I})}{T_i^\alpha}] = [\overset{(\mathrm{II})}{g}_{\alpha i}\overset{(\mathrm{II})}{T_j^\alpha} - \overset{(\mathrm{II})}{g}_{\alpha j}\overset{(\mathrm{II})}{T_i^\alpha}] \tag{1.3'}$$

where the symbol [ ] denotes value taken on the boundary $\Sigma : x^0 = 0$ between two adjacent regions (I) and (II). The symbol (N) above any quantity f signifies the value of f in the region (N), where $N = \mathrm{I, II}$. If the energy-momentum tensor is symmetric ($T_{ij} = T_{ji}$), then the conditions (1.1) and (1.3) together imply the condition (1.3').

In order to make the procedure for obtaining the above conditions mathematically complete, Synge (1960) later revised their approach by restricting the problem to a system of coordinates obtained from the system of admissible coordinates by a C^1 or C^2 transformation. However, as observed by Nariai, the revised procedure implies that the condition (1.1) must always hold without any physical justification. Moreover, it is no longer clear whether or not the condition (1.2) must hold. Hence Nariai (1965) reconsidered the problem of formulating suitable boundary conditions in general relativity, using a different method, namely a delta-function technique.

By stipulating that Einstein's field equations constructed from the combined metric tensor g_{ij} (defined by means of a step-function as a linear combination of two metric tensors specifying two adjacent regions) should be delta-singularities free, Nariai (1965) obtained the O'Brien-Synge (1952) boundary conditions (1.1) and (1.2). Furthermore, by invoking the consistency of Einstein's equations with the conservation law $T^i_{i;j} = 0$, he obtained the condition (1.3) as well as the following new one for the energy-momentum tensor:

$$[H^i_{j;i}] = \kappa\,[Q_j] \tag{1.4}$$

where

$$H_{ij} = E_{ij} - \tfrac{1}{2} g_{ij} E \tag{1.5}$$

and

$$E_{ij} = g^{lr} E_{lijr} \tag{1.6}$$

where E_{lijr} is an expression whose value on the boundary $\Sigma : x^0 = 0$ is given by

$$[E_{lijr}] = -(\tfrac{1}{4})[g^{mn}]\,[\Lambda_{(lr)m}\Lambda_{(ij)n} - \Lambda_{(lj)m}\Lambda_{(ir)n}] \tag{1.7}$$

where

$$\Lambda_{(ij)l} = \overset{(\mathrm{I})}{\Gamma}_{(ij)l} - \overset{(\mathrm{II})}{\Gamma}_{(ij)l} \tag{1.8}$$

and

$$\Gamma_{(ij)l} = g_{lr}\Gamma^r_{ij} \tag{1.9}$$

where Γ^i_{jl} is the usual affine connection and κ is the coupling constant.

The term Q_j in (1.4) is an expression whose value on the boundary is given by

$$[Q_j] = -(\tfrac{1}{4})[g^{mn}]\,[\Lambda_{(lm)n}S_j{}^l - \Lambda_{(jl)n}S_m{}^l] \tag{1.10}$$

where

$$S_m{}^l = \overset{(\mathrm{I})}{T_m{}^l} - \overset{(\mathrm{II})}{T_m{}^l} \tag{1.11}$$

The new condition (1.4) reduces to an identity if the condition (1.2) is replaced by the more stringent conditions

$$[\overset{(\mathrm{I})}{g_{ij,k}}] = [\overset{(\mathrm{II})}{g_{ij,k}}] \tag{1.12}$$

i.e. if the metric is of class C^1 (admissible coordinates). However, in the general case, owing to the rather complicated mathematical structure of the new condition (as can be seen in equations (A.1)-(A.4) of the Appendix), it has not been possible to prove directly that (1.4) is not merely an identity. On the other hand, physical problems in general relativity (such as 'collapse') have hitherto been based solely on some or all of the O'Brien-Synge conditions (1.1)-(1.3). Hence there is the need to investigate the status of the new condition (1.4) explicitly in various models in order to see its physical implications, if any.

In a previous paper (Kofinti, 1972) we have shown that both the Schwarzschild problem of determining the interior and exterior fields of a homogeneous statical sphere of perfect fluid, as well as Bonnor's (1954) approach to the stability of cosmological models, are consistent with the new boundary condition (1.4) without implying any new physical restrictions. As remarked by Bonnor, previous authors who studied the problem of stability reached conflicting conclusions partly as a result of using different initial conditions.

In another paper (Kofiniti, 1973), we considered a model consisting of a flat and a conformally flat manifold and showed that the condition may indeed be new and is not just an identity. However, it remains to find a physical situation in which the O'Brien-Synge conditions are valid but Nariai's new condition is not, or vice versa.

Now, the new condition (1.4) was derived solely on the basis of the usual Einstein field equations. Hence it is necessary to derive the appropriate form in the Einstein-Cartan theory. Although we cannot expect the new condition to be valid *a priori* in the Einstein-Cartan framework, the latter might serve as a discriminating ground between the new condition and the usual ones. Accordingly, in this paper, we investigate the new condition by means of the model we discussed recently (Kofinti, 1974) for the gravitational field of an isolated body embedded in Trautman-Kopczyński (1972) type of an expanding universe of spinning dust. We also investigate the validity and implications of the delta-function technique (employed in deriving the new condition) by applying it to the non-relativistic potential theory of a continuous distribution of gravitating matter.

Section 2 deals with the physical implications of the new boundary condition for our model. Section 3 contains an application of the delta-function technique to ordinary Newtonian potential theory, and Section 4 conclusions.

2. *Implications of the New Boundary Condition*

In a previous paper (Kofinti, 1974), we had, for the empty region (I) near an isolated body of mass m embedded in an expanding universe with spin, the metric

$$ds^2 = e^{\nu(r,t)}\,dt^2 - e^{\lambda(r,t)}\,dr^2 - \{K(t)\}^{-2/3} r^2\,d\Omega^2,$$
$$(d\Omega^2 \equiv d\theta^2 + \sin^2\theta\,d\varphi^2) \tag{2.1}$$

where, for $r \ll d$, $d^2 = \min(1/mK^{1/3} - 1,\ 1/2mK^{1/3})$,

$$e^{\nu(r,t)} = 1 - mK^{1/3}\{(2/r) - r^2 + 3\} \tag{2.2}$$

$$e^{\lambda(r,t)} = K^{-1/3}\{1 + (2mK^{1/3}/r) - (2mr^2/K^{1/3})\} \tag{2.3}$$

and $K(t)$ is the density of spin which is related to the cosmological radius $\mathscr{R}(t)$ by

$$|K(t)|\mathscr{R}^3(t) = 1 \tag{2.4}$$

The metric for the expanding universe of spinning dust (region (II)) surrounding the above region (I) is taken as

$$ds^2 = dt^2 - \{K(t)\}^{-2/3}(dr^2 + r^2\,d\Omega^2) \tag{2.5}$$

where

$$\tfrac{2}{3}K\ddot{K} - \dot{K}^2 + K^4 = 0 \tag{2.6}$$

which is a particular case of that in the Appendix of Trautman (1972). The spherical boundary surface Σ separating the adjacent regions (I) and (II) defined above is given by

$$\Sigma : x^0 \equiv r - 1 = 0 \tag{2.7}$$

using the notation of Section 1.

Using (2.1)-(2.3) and (2.5), we readily find that the conditions (1.1) and (1.2) are satisfied on Σ at the epoch $t = t_0$ such that $K(t_0) = 1$. Now, since the metrics are not of class C^1, we expect Nariai's new condition (1.4) to be non-trivial in our model. Accordingly, we proceed to analyse the implications of the new condition at the epoch $t = t_0$.

From (2.1)-(2.3) and (2.5), we find by computation that the only non-vanishing components of $\Lambda_{(ijk)}$ on the boundary $\Sigma : x^0 \equiv r - 1 = 0$ are

$$[\Lambda_{(11)0}] = -[\Lambda_{(10)1}] = \tfrac{1}{6}(1 + 4m)[\dot{K}]$$
$$[\Lambda_{(11)1}] = 3m \tag{2.8}$$

using the notation of Section 1. We find also that the only non-vanishing components of E_{ij} on the boundary are:

$$[E_{00}] = -[E_{11}] = \tfrac{1}{144}(1 + 4m)^2[\dot{K}^2] \tag{2.9}$$

and hence

$$[H_{ij}] = 0 \tag{2.10}$$

On the other hand, we find that Q_j has one non-vanishing component given by

$$[Q_0] = -\frac{2\pi}{9}(1+4m)[\dot{K}][\ddot{K}-\dot{K}^2] \tag{2.11}$$

Hence, from (2.10) and (2.11), the new condition (1.4) implies

$$[\dot{K}][\ddot{K}-\dot{K}^2] = 0 \tag{2.12}$$

which, in view of (2.6) with $[K] = 1$ reduces to

$$[\dot{K}][\ddot{K}-3] = 0 \tag{2.13}$$

Now, if $[\dot{K}] = 0$, the condition (1.4) clearly reduces to an identity, in view of (2.10) and (2.11). Thus the new condition applies non-trivially only when

$$[\dot{K}(t_0)] \neq 0 \qquad \text{and} \qquad [\ddot{K}(t_0)] = 3 \tag{2.14}$$

Thus the new condition gives rise to a restriction on the rate of spinning of the particles lying on the boundary. In view of (2.4) and (2.14), the new condition is also a boundary restriction on the deceleration of the expanding universe of spinning particles. In this case, the first condition in (2.14) is merely a statement about the non-vanishing of the Huble parameter at the epoch t_0.

On the other hand, we find that in our model the condition (1.3) implies that

$$[2\ddot{K} - 3\dot{K}^2] = 0 \tag{2.15}$$

which is clearly incompatible with (2.6) for $K(t_0) = 1$. Hence, in our model, Nariai's new condition (1.4) is satisfied but the O'Brien–Synge condition (1.3) is not. We have therefore shown that the new condition (1.4) is indeed independent of the usual one in (1.3).

3. *Non-Relativistic Considerations*

Let $x^\alpha(\alpha = 1, 2, 3)$ be a system of rectangular Cartesian coordinates relative to which the boundary Σ between two adjacent regions (I) and (II) of a continuous distribution of gravitating matter is specified by

$$\Sigma : x^0 = 0 \tag{3.1}$$

where x^0 is any one of the three coordinates $x^\alpha(\alpha = 1, 2, 3)$. Let

$$\overset{(N)}{\varphi}(x^\alpha, t) \qquad \text{and} \qquad \overset{(N)}{\rho}(x^\alpha, t)$$

be, respectively, the Newtonian gravitational potential and density corresponding to the region $N(= \text{I}, \text{II})$.

Define now the combined gravitational potential φ in the domain I + II by

$$\varphi(x^\alpha, t) = \sum_{N=\mathrm{I}}^{\mathrm{II}} \overset{(N)}{\varphi}(x^\alpha, t)\theta_N \tag{3.2}$$

where

$$\theta_{\mathrm{I}} = \theta(x^0), \qquad \theta_{\mathrm{II}} = \theta(-x^0) \tag{3.3}$$

and $\theta(x^0)$ is the step-function given by

$$\theta(x^0) = \begin{cases} 1 & (x^0 > 0) \\ \frac{1}{2} & (x^0 = 0) \\ 0 & (x^0 < 0) \end{cases} \tag{3.4}$$

The field equation for region (N) is

$$\overset{(N)}{\varphi}_{,\alpha\alpha} = -4\pi G \overset{(N)}{\rho} \tag{3.5}$$

where G is the gravitational constant, and that for the combined region I + II is

$$\varphi_{,\alpha\alpha} = -4\pi G\rho \tag{3.6}$$

where ρ is the density in the combined region. From (3.2) we obtain

$$\varphi_{,00} = (\overset{(\mathrm{I})}{\varphi} - \overset{(\mathrm{II})}{\varphi})\delta'(x^0) + 2(\overset{(\mathrm{I})}{\varphi}_{,0} - \overset{(\mathrm{II})}{\varphi}_{,0})\delta(x^0) + \sum_{N=\mathrm{I}}^{\mathrm{II}} \varphi_{,00}\theta_N$$

$$\varphi_{,\alpha\alpha} = \sum_{N=\mathrm{I}}^{\mathrm{II}} \varphi_{,\alpha\alpha}\theta_N, \qquad (\alpha \neq 0) \tag{3.7}$$

Hence the necessary and sufficient conditions for $\varphi_{,\alpha\alpha}$ to be free from any kind of δ singularities are

$$[\overset{(\mathrm{I})}{\varphi}] = [\overset{(\mathrm{II})}{\varphi}] \tag{3.8}$$

$$[\overset{(\mathrm{I})}{\varphi}_{,0}] = [\overset{(\mathrm{II})}{\varphi}_{,0}] \tag{3.9}$$

where the symbol [ ] denotes value taken on the boundary $\Sigma : x^0 = 0$. The conditions (3.8) and (3.9) derived above by the delta-function technique are just the usual ones in Newtonian potential theory, namely, that the potential and its derivative normal to the boundary Σ are continuous across the boundary. Physically, the condition (3.9) implies that a test particle lying on Σ is not accelerated in the direction normal to Σ.

We note also that the field equation (3.6) in the combined region I + II implies

$$-4\pi G\rho = -4\pi G \sum_{N=\mathrm{I}}^{\mathrm{II}} \overset{(N)}{\rho}\,\theta_N \tag{3.10}$$

in view of (3.7)–(3.9). Thus, unlike the relativistic case, ρ is a linear combination of $\overset{(N)}{\rho}$ ($N = \mathrm{I}, \mathrm{II}$). Hence, as one would expect, the non-linear relativistic effect arising from the H-term in (1.4) disappears in Newtonian theory.

Now, in each of the regions (N), the equation of continuity holds, i.e.,

$$\partial \overset{(N)}{\rho}/\partial t + (\overset{(N)}{\rho}\overset{(N)}{u_\alpha})_{,\alpha} = 0 \qquad (N = \mathrm{I}, \mathrm{II}) \tag{3.11}$$

where $\overset{(N)}{u_\alpha}(x^\alpha, t)$ is the 3-velocity of fluid in region (N). Let the 3-velocity $u_\alpha(x^\alpha, t)$ in the combined region I + II be defined by

$$u_\alpha = \sum_{N=\mathrm{I}}^{\mathrm{II}} \overset{(N)}{u_\alpha}\theta_N \tag{3.12}$$

Then, from the equation of continuity in the combined region I + II, namely

$$\partial\rho/\partial t + (\rho u_\alpha)_{,\alpha} = 0 \tag{3.13}$$

and (3.10), we obtain

$$\begin{aligned}\sum_{N=\mathrm{I}}^{\mathrm{II}} &\{(\partial \overset{(N)}{\rho}/\partial t)\theta_N + (\overset{(N)}{\rho}\overset{(N)}{u_\alpha})_{,\alpha}\theta_N^2\} + (\overset{(\mathrm{I})}{\rho}\overset{(\mathrm{I})}{u_0} - \overset{(\mathrm{II})}{\rho}\overset{(\mathrm{II})}{u_0})\delta(x^0)\\ &+ 2(\overset{(\mathrm{I})}{\rho}\overset{(\mathrm{I})}{u_0}\theta_{\mathrm{I}} - \overset{(\mathrm{II})}{\rho}\overset{(\mathrm{II})}{u_0}\theta_{\mathrm{II}})\delta(x^0) + (\overset{(\mathrm{I})}{\rho}\overset{(\mathrm{II})}{u_\alpha} + \overset{(\mathrm{II})}{\rho}\overset{(\mathrm{I})}{u_\alpha})_{,\alpha}\theta_{\mathrm{I}}\theta_{\mathrm{II}}\\ &+ (\overset{(\mathrm{I})}{\rho}\overset{(\mathrm{II})}{u_0} + \overset{(\mathrm{II})}{\rho}\overset{(\mathrm{I})}{u_0})(\theta_{\mathrm{II}} - \theta_{\mathrm{I}})\delta(x^0)\\ &= 0\end{aligned} \tag{3.14}$$

which, in view of (3.11) reduces on the boundary $\Sigma : x^0 = 0$ to

$$\{(\overset{(\mathrm{I})}{\rho} - \overset{(\mathrm{II})}{\rho})(\overset{(\mathrm{I})}{u_\alpha} - \overset{(\mathrm{II})}{u_\alpha})\}_{,\alpha} - 8(\overset{(\mathrm{I})}{\rho}\overset{(\mathrm{I})}{u_0} - \overset{(\mathrm{II})}{\rho}\overset{(\mathrm{II})}{u_0})[\delta(x^0)] = 0 \tag{3.15}$$

Hence the following boundary conditions must be satisfied separately:

$$[\overset{(\mathrm{I})}{\rho}\overset{(\mathrm{I})}{u_0}] = [\overset{(\mathrm{II})}{\rho}\overset{(\mathrm{II})}{u_0}] \tag{3.16}$$

$$[\{(\overset{(\mathrm{I})}{\rho} - \overset{(\mathrm{II})}{\rho})(\overset{(\mathrm{I})}{u_\alpha} - \overset{(\mathrm{II})}{u_\alpha})\}_{,\alpha}] = 0 \qquad (\alpha = 1, 2, 3) \tag{3.17}$$

in order to avoid δ singularities.

The condition (3.16) is just the usual one which expresses the continuity of the normal component of 3-current. Indeed, the conditions (3.8), (3.9) and (3.16), are, respectively, the non-relativistic analogues of (1.1)-(1.3). However, the condition (3.17) appears to be a new one which is the non-relativistic analogue of (1.4). Denoting the discontinuities on the boundary of density and 3-velocity by $\Delta\rho$ and Δu_α, respectively, (3.17) takes the form

$$[(\Delta\rho\,\Delta u_\alpha)_{,\alpha}] = 0 \qquad (\alpha = 1, 2, 3) \tag{3.18}$$

Introducing the rate-of strain matrix $\boldsymbol{\sigma}$ by

$$\sigma_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \tag{3.19}$$

we can also write condition (3.17) in the form

$$[\Delta\rho\,\Delta\,\mathrm{Tr}\,\boldsymbol{\sigma} + (\Delta\rho)_{,\alpha}\Delta u_\alpha] = 0 \tag{3.20}$$

where Tr denotes trace.

For an incompressible fluid (ρ = constant), or in the case $\Delta u_\alpha = 0$, we see that (3.20) is a condition on the rate-of-strain of the medium on the boundary $\Sigma : x^0 = 0$. On the other hand, if the fluid is compressible and has an isentropic equation

$$p = k\rho^\gamma \tag{3.21}$$

where p is the pressure and k, γ are constant, then (3.20) is a boundary condition relating pressure-gradient and the rate-of-strain.

4. *Conclusions*

The above analysis suggests that the new general relativistic boundary condition (1.4) may not be a mere identity. In our model, the new condition relates to the rate-of-spin of the particles on the boundary at an epoch. This in turn implies a non-vanishing Hubble parameter and a boundary condition on the deceleration of the universe. The discussion also reveals that the new condition is entirely of a different physical nature from the usual ones employed in general relativity.

Whenever possible, it is instructive to examine the implications of general relativistic results in the framework of ordinary Newtonian theory. In this spirit, our analysis shows first of all that the delta-function technique employed in general relativity is physically reasonable at the Newtonain level. (Indeed, we have also checked that the technique does yield the usual boundary conditions in the non-relativistic Maxwell electromagnetic theory (Kofinti, unpublished).) In addition to yielding the usual boundary conditions we obtained the new one given by (3.20). The latter, which may be regarded as the non-relativistic analogue of (1.4), is a condition on the boundary behaviour of rate-of-strain and pressure gradient in the case of a compressible fluid with an isentropic equation. The new condition is therefore likely to be relevant in thermodynamic considerations in gas dynamics, especially in an adiabatic flow.

The above non-relativistic considerations also throw some light on why the new relativistic condition (1.4) holds identically in situations like the Schwarzschild problem (Kofinti, 1972) where the boundary surface is pressure free.

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Appendix

The expression on the left-hand side of the new condition (1.4) is given in full by

$$
\begin{aligned}
[H_{m}{}^{r}{}_{;r}] = \sum_N \Big[& \overset{(N)}{\Gamma}_{(rc)d}\{\overset{(N)}{A}{}^{s}{}_{.m.s..}^{\;\,r\;\;(dc)} - (\tfrac{1}{2})\delta_m{}^r \overset{(N)}{A}{}^{s}{}_{.n.s..}^{\;\,n\;\;(dc)}\}\Big] \\
& - (\tfrac{1}{4})\Big[\sum_N \overset{(N)}{\Gamma}_{(rc)d}\Big]\Big[\sum_N \{\overset{(N)}{A}{}^{s}{}_{.m.s..}^{\;\,r\;\;(dc)} - (\tfrac{1}{2})\delta_m{}^r \overset{(N)}{A}{}^{s}{}_{.n.s..}^{\;\,n\;\;(dc)}\} \\
& + \{B^{s}{}_{.m.s..}^{\;\,r\;\;(dc)} - (\tfrac{1}{2})\delta_m{}^r B^{s}{}_{.n.s..}^{\;\,n\;\;(dc)}\} \\
& - \sum_N \{\overset{(N)}{A}{}^{(d}{}_{.m...a}^{\;\,|r|c)a} - (\tfrac{1}{2})\delta_m{}^r \overset{(N)}{A}{}^{(d}{}_{.n...a}^{\;\,|n|c)a}\} \\
& + \{B^{(d}{}_{.m...a}^{\;\,|r|c)a} - (\tfrac{1}{2})\delta_m{}^r B^{(d}{}_{.n...a}^{\;\,|n|c)a}\} \\
& - (\tfrac{1}{2})g^{rc} \sum_N \overset{(N)}{A}{}^{s}{}_{.m.s.a}^{\;\,d\;\;a} - B^{s}{}_{.m.s.a}^{\;\,d\;\;a}\}\Big] \\
& -(\tfrac{1}{4})\{\delta_m{}^{m'}[g^{rn'}] - (\tfrac{1}{2})\delta_m{}^r[g^{m'n'}]\}[g^{ls}g^{ab}] \\
& \times \Big[\sum \overset{(N)}{A}_{lm'n'sab,r} - B_{lm'n'sab,r}\Big]
\end{aligned}
\qquad (A.1)
$$

where (ab) denotes $(ab + ba)/2$, and

$$\overset{(N)}{A}_{lmnsab} \equiv \overset{(N)}{\Gamma}_{(ls)a} \overset{(N)}{\Gamma}_{(mn)b} - \overset{(N)}{\Gamma}_{(ln)a} \overset{(N)}{\Gamma}_{(ms)b} \tag{A.2}$$

and

$$\begin{aligned} B_{lmnsab} \equiv & \overset{(\mathrm{I})}{\Gamma}_{(ls)a} \overset{(\mathrm{II})}{\Gamma}_{(mn)b} + \overset{(\mathrm{II})}{\Gamma}_{(ls)a} \overset{(\mathrm{I})}{\Gamma}_{(mn)b} \\ & - \overset{(\mathrm{I})}{\Gamma}_{(ln)a} \overset{(\mathrm{II})}{\Gamma}_{(ms)b} - \overset{(\mathrm{II})}{\Gamma}_{(ln)a} \overset{(\mathrm{I})}{\Gamma}_{(ms)b} \end{aligned} \tag{A.3}$$

The right-hand side of (1.4) is given in full by

$$\begin{aligned} \kappa [Q_m] = (\tfrac{1}{4}) \Bigg[& \sum_N \Big(\overset{(N)}{\Gamma}{}^a_{na} \overset{(N)}{R}_m{}^n - \overset{(N)}{\Gamma}{}^a_{mn} \overset{(N)}{R}_a{}^n \Big) \\ & - \overset{(\mathrm{I})}{\Gamma}{}^a_{na} \overset{(\mathrm{II})}{R}_m{}^n - \overset{(\mathrm{II})}{\Gamma}{}^a_{na} \overset{(\mathrm{I})}{R}_m{}^n \\ & + \overset{(\mathrm{I})}{\Gamma}{}^a_{mn} \overset{(\mathrm{II})}{R}_a{}^n + \overset{(\mathrm{II})}{\Gamma}{}^a_{mn} \overset{(\mathrm{I})}{R}_a{}^n \Bigg] \end{aligned} \tag{A.4}$$

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